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THE NATURE OF SPACE—I

INTRODUCTION

The topic of the present study is to be understood in a very restricted sense, and the essential restrictions must be made plain at the outset. Every textbook of geometry may be said to be an essay on the nature of space, that is to say, of space in the abstract—space *assumed* either as matter of fact or as matter of hypothesis. This little essay does not pretend to rival the textbooks of geometry. It does, indeed, have occasion to treat of some very elementary geometrical concepts, but not of anything more complicated than the straight line. At the same time, it is not content to assume abstract space, either as matter of fact or as hypothesis. On the contrary, its princiapl object is to exhibit the conception of space in its actual setting in experience.

However, it is not the intention to try to give an account of the whole of this setting, but only of so much of it as is strictly relevant to the *geometrical* treatment of space. With the further physical aspects of space we shall have almost nothing to do. The notion of mass and the notion of a measurable duration will not enter into our discussion—not to speak of the notions of force and energy. But a very important aspect of the meaning of *distance* consists in its relations to mass, duration, *etc.* No man can throw a baseball two hundred yards; and no man can run a mile in three minutes. From the geometrical point of view, the kilometer has no properties distinct from those of the millimeter; and if all the distances in the world were multiplied by a million there would be no change at all. From the physical point of view, even the doubling of all lengths and distances would have a very profound effect upon the world, if bodies on the surface of the earth continued to fall about sixteen feet during the first second.

The reader will, therefore, realize that in restricting consideration to the geometrical conception of space, we very greatly simplify our problem, and at the same time seriously limit the possible value of any solution which we may reach. However successful we may be, we shall have taken but a single step toward the larger systematic knowledge of space, which can only be obtained as an essential part of the knowledge of the general character of the physical world.

There are two motives for this restriction of the inquiry. One is purely personal: the narrowness of the writer's knowledge and competence. This, if it stood alone, would be a better reason for not publishing at all than for publishing the result of a truncated investigation. But there is a further motive in the belief that within the limits that are thus laid down a tolerably complete and satisfactory solution of the problem can be given.

There is this also to be said. While results, such as are here given, are seriously limited in their range, they have a place of their own in the system of science; and at the present juncture their importance may be very great. How far Professor Einstein has gone into detail in this matter, I do not know; but it is clear from his account of the use of "measuring-rods" that some such theory as that here set forth is presupposed by him in his two-fold theory of relativity. Professor Whitehead, who is the author both of an independent account of the electromagnetic theory of relativity and of a revision of Newton's theory of gravitation, has set forth very fully his conception of spatial order and measurement in two remarkable volumes, *An Enquiry into the Principles of Natural Knowledge* and *The Concept of Nature.* That conception is radically inconsistent with the analysis here given.

Certain of the differences between Professor Whitehead's account and the present one may be specified as follows. His account starts with *events;* the present account starts with a certain class of *objects,* namely, *physical solids.* Among the events which he assumes, some are unlimited in three dimensions; here only finite solids are assumed. The assumed unlimited events have, with reference to their fourth (temporal) dimension, a definite *shape;* they are analogous to the three-dimensional space between parallel planes; the solids from which the present account starts are of all possible shapes, but no one shape is assumed as given. Most important of all, however, is the fact that Professor Whitehead makes the conceptions of the point, line, and surface—and the solid, too, for that matter—as well as of linear order, length, and distance, logically dependent upon the intersection of moments of different time-systems; while to me this appears to be a most unfortunate distortion of the actual system of relationships.

There is a deeper ground of difference, however, which ought not to pass unnoticed. Professor Whitehead's work is based upon a certain theory as to the nature of experience, which seems to me to be extremely doubtful. He takes his start from certain data, which, as he believes, are given to us in sensuous experience. My own point of departure is in the behavior of things toward one another, as we manipulate them.

Among Professor Whitehead's incontestably important contributions is his method of "extensive abstraction." In the precise form in which he has described it, it was not available for my purposes. But, in an appendix, I have used a simplified, and in some respects strengthened, form of the method for the definition of the point, the line, and the surface as sets of solids; and I have at the same time showed how the method is related to my own assumptions.

The main body of the essay is in two parts. The first is mathematical; but, lest this should affright any modest philosophical reader, let me hasten to add that the mathematics is of the very simplest character—quite as easy as the easiest pages of the first book of Euclid. (In the notes there are passages that may make rather more difficult reading; but these are not essential.) The object of this part is to lead to as clear as possible a conception of certain of the more fundamental geometrical entities and relations.

The second part is physical. Its object is to determine the empirical foundations of geometry (so far as these may properly be held to lie within the limits of a physical, rather than a psychological, inquiry). The point and the peculiar relation between points, which, in the mathematical part, figure as primary assumptions in terms of which explanation is to be made, are here themselves the goal that is to be attained. The physical part of the inquiry is thus directly supplementary to the mathematical. The former owes its problem to the outcome of the latter.

At the end of the second part, an opportunity arises for making a suggestion the import of which extends far beyond the subject of these pages. This opportunity is afforded, first, by the close similarity of the conception here reached of the principles of geometry, to the conception of physical principles long ago advanced by Galileo, as well as to that recently advanced by Poincaré; and, secondly, by a further analogy to the general principle of empirical science—the uniformity of nature. I shall take advantage of this opportunity to the extent of offering the briefest possible comment.[1]

[1] With the exception of the appendix dealing with the method of extensive abstraction, this essay has lain in manuscript for the last five years. An earlier, more general study, *The Nature of Primary Qualities*, was published in the *Philosophical Review* for September, 1913 (Vol. XXII, pp. 502ff.). A second paper, *On the Distinction between Primary and Secondary Qualities*, dealing more fully with certain outlying epistemological questions, appeared in this JOURNAL for February 28, 1918 (Vol. XV, pp. 113ff.).

THE FUNDAMENTAL CONCEPTS OF GEOMETRY

Every reader, the memory of whose youthful days has not faded out completely, will recall that in the study of elementary geometry one sets out from a body of concepts that are accepted as too simple and clear to need definition, and that in terms of these primary concepts one defines all other concepts that belong to the science. The number of these assumed "indefinables" is usually very great; but, since they are not plainly listed and set apart, they may easily seem to be far fewer than a careful search would show.

It has long been an enterprise of mathematicians, dating particularly from the researches of Leibniz, first, to make an accurate list of the assumed concepts of geometry, and, secondly, to reduce their number to the absolute minimum by defining all that can be defined. This enterprise has in recent years been rewarded with a large measure of success. It has been found possible to base the concepts of geometry upon two "indefinables," one of which is an entity, the other a relation.[2]

What was not anticipated in the old days is the fact that a considerable freedom of choice is possible in choosing the indefinables. The entity chosen is, indeed, usually the *point*. But for the relation there are three important alternatives to choose from, giving rise to three fairly distinct types of geometrical system: projective geometry, descriptive geometry,[3] and metrical geometry.

(i) For projective geometry the indefinables are now usually "point" and a certain relation between three points called "to be

[2] This last statement must not be misunderstood. Geometry makes the freest use of the concepts of formal logic, as well as of the more complex mathematical concepts (particularly those of arithmetic) which are definable in terms of logical concepts. Such terms as "if—then," "and," "or," "not," "any," "such as," "identical with," *etc.;* as well as "one," "two," "as many," "more," "twice as many," *etc.*, are thickly sprinkled all over the pages of our geometries. Some of these must, in any system of logic, be accepted as indefinable. When, therefore, we speak of the two indefinables of a system of geometry, we mean the two *peculiar* indefinables, assumed in addition to the omnipresent concepts of logic.

To guard against misplaced verbal criticism, it may be added that there is an interpretation of geometry, according to which its "indefinables" are really defined—namely, by the "axioms," or "postulates," from which the demonstrations of the science proceed. The argument of the present chapter is unaffected by these considerations.

[3] The usage of this term is unfortunately inconsistent. In the following pages it will be used to denote the type of geometry for which measurement is a wholly secondary matter, but which assumes from the outset—as projective geometry does not—the conception of the *order* of three points in a straight line. *Cf.* B. Russell, *The Principles of Mathematics,* p. 382 and Chap. XLVI; L. Couturat, *Les Principes des Mathématiques,* p. 142 and pp. 159ff.

collinear.'' In common language this relation would be expressed by saying that the three points were in one straight line. The same mode of expression may, indeed, be used by the mathematician; but he has then three indefinables instead of two: namely, ''point,'' ''line'' (or ''straight line''), and ''to be in.'' It is simpler to start with ''point'' and ''collinear,'' and to define the straight line as a peculiar set of points; that is to say, as the set of all the points that are collinear with two distinct points, together with the two points themselves.

We may pause here to emphasize the fact that collinearity is a relation between *three* points. The relations with which logic has for the most part dealt are relations between two terms; and from this fact a wide-spread prejudice has arisen, to the effect that all relations are confined to two terms. But the case of collinearity itself is clear evidence to the contrary. *Any two* points are connected by a straight line; but when this is true of three points it constitutes a certain definite relation between them.

It is true, that instead of regarding collinearity as a relation between three points, we may equally well regard it as a *quality of a collection* of three points.[4] That is, of course, because the relation of collinearity is symmetrical. If A is collinear with B and C, it is collinear with C and B; and furthermore B is collinear with A and C, and C is collinear with A and B. Such a relation can always be regarded as a quality that is predicable of the collection as a totality. Instead of saying that Henry is a cousin of Stephen or that Stephen is a cousin of Henry, I may equally well say that Henry and Stephen are a pair of cousins.

(ii) For descriptive geometry the approved indefinables are ''point'' and ''to be between.'' This latter is, again, a relation between three points. It would be expressed in ordinary language by saying that one point was in a line with two others, and between them; but in descriptive geometry it is accepted as a simple, inexplicable *datum*.

The between-relation differs from the relation of collinearity in being symmetrical only with respect to two of the three points and asymmetrical with respect to either of those two points and the

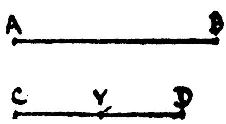
[4] The properties that we ascribe to things are of two sorts: first, those that we may ascribe to a single thing and, secondly, those that may only be ascribed to two or more things conjointly. ''To be brave'' is an example of the first sort; ''to be a cousin of'' is an example of the second sort. These two sorts of properties may, with essential fidelity to tradition, be distinguished as ''qualities'' and ''relations.'' It should be observed, however, that in accordance with this usage, while ''to be a cousin of'' is a relation, ''to be a cousin of Peter'' is a quality; for we may affirm it of a single subject.

third point. If A is between B and C, then it is between C and B; but B is not between A and C, nor is C between A and B.[5] Accordingly the between-relation is not interpretable as a quality of the collection of three points. The peculiar position of the middle point can not be expressed in any such way. The relation may, however, be regarded, not as a relation between three terms, but as a relation between two; namely, the middle point and the collection consisting of the other two points.

(iii) Metrical geometry has been less successful than the other two types of geometry in finding appropriate and serviceable indefinables. Metrical geometry is what most of us mean when we use the word without an adjective—the science as set forth by Euclid and his direct successors. In fact the expression "metrical geometry" is etymologically a crude tautology. But the science of space has been so extended during the last century, that its original subject of spatial measurement has become for it an altogether secondary interest. The historical fact remains, that metrical geometry is the original type, and that projective and descriptive geometry are comparatively recent specialized developments of conceptions first reached by the metrical mode of approach. The less satisfactory condition of the foundations of metrical geometry must, therefore, be deeply regretted by those who wish to understand the place of geometry in human experience.

Since the time of Leibniz, the generally preferred indefinables for metrical geometry have been "point" and a *class of relations* between two points, called "distances." It is assumed that the same relation of distance that subsists between two points may also subsist between two other points. As a matter of fact the distance between two points never appears in a proposition except as it is in some way compared with the distances between other points.

The unsatisfactoriness of these indefinables depends upon the awkwardness of the procedure by which it is necessary to introduce the notion of one distance being *greater than* another distance. The following device is as simple as any:



"The distance AB is greater than the distance CD" means that there exists a point Y, such that the distance CY is the same as the distance DY, and such that there is no point X such that the distances AX, BX, and CY are all the same.[6]

[5] Sometimes a certain class of exceptions is admitted: it is assumed that any point is "between" itself and any other. (In the same way it may be assumed in projective geometry that any point is collinear with itself and any other point.) This is to be regarded as a merely verbal matter, to be determined by convenience of terminology.

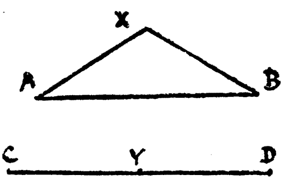
It is to be observed that this device does not enable us to define the relation "greater than" except on the implied assumption that space is of at least two dimensions. If we limit space to a single straight line, then, if the distances AB and CD are not identical, AB is *necessarily* greater than CD by the above definition; for, on that supposition, if the distances CY and DY were the same, we could *never* find a point X such that AX, BX, and CY were identical.

If we assume that space is of only one dimension, we can, indeed, define the relation "greater than" with reference to any two distances that are *commensurable* with each other. We say, for example, that if the distances AB, BC, CD, and DE are all the same, then the distance AE *contains* the distance AB four times. If, in the same way, the distance XY contains AB three times, then, since (in the arithmetical sense of the terms) four is *greater than* three, we say: "AE is greater than XY."[7]

How unfortunate it is that the relation "greater than" should be made to depend upon multidimensionality or else upon commensurability, is seen from the consideration of *time*. Time we are in the habit of conceiving on the analogy of the straight line, as a space of one dimension. Now, in the case of time, we make no scruple to assume that one interval may be longer than another, altogether independently of the question of commensurability. Why should we have to look outside of the straight line itself in order to give a general meaning to the proposition that one of its segments is greater than another?[8]

When we reflect upon the matter, we can see that there is at

[6] This figure illustrates the fact that if AB were less than CD a point X that satisfied the given requirement could easily be found. It must, however, always lie outside of the line AB.



[7] It might be supposed that this mode of definition could be extended to incommensurable distances by the method of limits; but this appears to be impossible. One can not even give a meaning to the expression, "B lies between A and C," unless AB and BC are commensurable.

[8] It is, of course, possible to assume "greater-than" (in the sense of a relation between two distances) as a third geometrical indefinable, and this is sometimes, openly or covertly, done. But such a procedure, *if it is not unavoidable*, has this demerit: it puts a limitation upon the understanding of the subject-matter of the science—it effectually prevents a maximum clearness. It is true that a high degree of simplicity is thereby facilitated; but in the study of the principles of mathematics simplicity is too dearly purchased at the expense of clearness. Each additional undefined term that might have

bottom a very good reason why the distance-relation makes a very poor conception upon which to base the science of spatial measurement. As we have already remarked, one distance-relation never appears alone in a proposition, but only in comparison with another distance-relation. Geometry takes cognizance of no peculiar properties of any particular distance. Hence the distance between two given points, when it is considered by itself, apart from any relation to the distances between other points, is as nearly as possible an *empty* concept.

This may, perhaps, be made clearer by an illustration. Suppose a man cast ashore upon a desert island, where among the useful articles that he has saved from the wreck he finds a pair of compasses and a straight-edge, but no yard-measure and nothing of any known ratio to the yard. Can he reproduce the yard? He can not. The yard, like every other measure of length, is a *conventional* unit. To be a yard long means to be just as long as something else accepted as being a yard long. If a standard of reference is wanting, the yard disappears.

The suggestion thus arises that for the construction of a metrical geometry the fundamental relation ought to be a relation between *four* points, or, if you please, between two "pairs" of points —collections of points consisting of two each.[9] One such relation is "to be just as far apart as." Another is "to be farther apart

been dispensed with means so much more superficiality—it means that the labor of analysis that ought to be done has not been done.

In this connection the remark may be made, that the number of assumed axioms—provided these are mutually independent and are sufficient to prove what they are expected to prove—is almost entirely indifferent. If anything, the greater number of axioms is preferable, for this may show a finer analysis; but this does not appear to be necessarily the case. There is a good deal of popular misunderstanding on this point, because of the fact that so much endeavor has been directed by mathematicians upon the elimination of *superfluous* axioms—axioms that could really be proved from the remaining assumptions. It needs to be clearly understood that if none of the axioms of a given set is superfluous—if they are all mutually independent—their number is, generally speaking, a matter of no importance.

[9] To be perfectly precise, one must admit the limiting case in which the "pair" consists of but one and the same point "twice considered." This, however, is a topic that belongs to general logic. The pair "A and B" may there be defined as the class (or collection) of which A is a member and B is a member, and which has no other members. It is not specified, and it is not implied, that A and B must be distinct.

The classical geometry conceived of this limiting case in a characteristically different way. Instead of "considering the same point twice over," it admitted *coincident* points. There is no reason why we today should not adopt a similar procedure; and for the purposes of this study there would be some advantages in doing so. I have preferred, however, to conform to the prevailing fashion as far as possible.

than." There are several reasons for preferring the latter, of which only one is strictly pertinent in this place; namely, that the choice of the former amounts to practically the same thing as the assumption of distance itself as an indefinable class of relations.[10]

One other reason, however, is of too great methodological importance to be denied mention. The assertion of a relation such as "to be just as far apart as" is generally based, not on positive evidence, but on the absence of negative evidence. If we are dealing with physical objects, and A and B are farther apart than C and D, we can in many instances attest this fact with a high degree of assurance. As, however, this relation *approaches* the condition where A and B are just as far apart as C and D, the certainty of our judgment decreases; and a condition is finally reached where all that we can say is that we see no further difference. Generally speaking, propositions asserting the relation "to be farther apart than" are of a higher degree of probability than those asserting the relation "to be just as far apart as." What, therefore, the proposition, that A and B are just as far apart as C and D, actually *means* in the system of science is that, within the limits of our observation, A and B are not farther apart than C and D, and C and D are not farther apart than A and B.

It is very striking, how, when the relation, "to be farther apart than," is taken as fundamental, the difficulties dwindle away. This can best be shown by exhibiting a specimen series of definitions. It is true that, from the mathematical standpoint, the definitions that occur in mathematics are matters of merely verbal significance. They are statements that certain newly presented symbols may, at will, be substituted for certain combinations of previously presented symbols. Nevertheless, it is in the definitions that the analysis of the complex subject-matter of the science most clearly appears.

The series of definitions given in the following pages extends as far as the introduction of the notions which are fundamental to projective and descriptive geometry. Farther than that we need not go; for the "defining-value" of these notions is well known, and that of our own indefinables will have been shown to be at least

[10] It may be recalled that in Professor Veblen's well-known account of the foundations of elementary geometry *two* indefinable relations are employed. In addition to the relation of *congruence*, which is essentially the same as "to be just as far apart as," he introduces the relation of *order* between three points, which is essentially the same as the between-relation. (I say "essentially," because in each case there may be an unimportant technical difference.) The resulting system is, therefore, not purely metrical, but is a mixture of metrical and descriptive elements; and it labors under the disadvantage which was pointed out in a previous note.

as great. Our task, then, will be to make clear what, from the metrical standpoint, is meant by the *order of points in a straight line.*

Indefinables

I. Point.

II. To be farther apart than.

Points are to be represented by capital letters. Different letters need not indicate distinct points.

Definitions

I. If A and B are not farther apart than C and D, and C and D are not farther apart than A and B, then A and B are said to be *just as far apart as* C and D.

II. The *distance* AB is the class of pairs of points that are just as far apart as A and B.

This definition calls for some comment. Instead of defining a distance as a *class* of pairs of points, we might define it as the *distinguishing property* of that class. That is to say, we might define the distance AB as the property of being just as far apart as A and B. This would be in closer accord with our common-sense notion of distance. But, as students of logic well know, propositions about classes, and propositions about the distinguishing properties of classes, run parallel to each other. The two sorts of propositions represent two different ways of regarding the same facts—in *extension* and in *intension,* to use the traditional terms. It is the general custom of mathematicians to prefer the extensive treatment; and there is no reason why we should do otherwise.

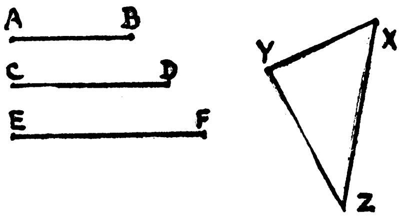
The definition of a distance as a class has this consequence: that we shall have to speak of *the same distance,* not of *equal* (or *equivalent*) *distances.* According to our definition, if A and B are just as far apart as C and D, the distance AB and the distance CD are identical. If we had chosen to define a distance as a property, the case would be different. The property of being just as far apart as A and B is not identical with the property of being just as far apart as C and D. The relation between these properties is that they *imply each other,* or are equivalent.

It is further to be remarked that the distance AB may be defined as the (symmetrical) relation subsisting between *two* points, by reason of the fact that they are just as far apart as the points A and B. This definition has nothing in particular to recommend it. But it is worth while to reflect upon it a little, because of the way in which it brings to the surface the unfitness of the two-term distance relation to serve as one of the bases of geometry.

III. If A and B are farther apart than C and D, the distance AB is said to be *greater than* the distance CD.

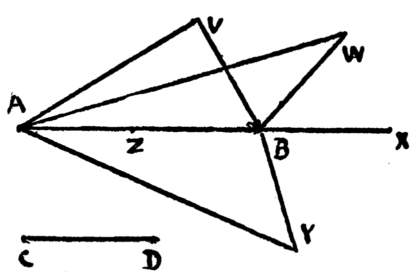
Here it is to be observed that the phrase "to be greater than" is defined in a particular technical sense: namely, as it is applied to distances. In arithmetic a relation having the same name is defined with reference to real numbers; and it is one of the first tasks of metrical geometry to justify this ambiguous use of the words by showing that its own greater-than relation has the same formal properties as the arithmetical relation. For it is on this identity of the formal properties of the arithmetical and the geometrical greater-than, that the theory of spatial measurement rests.

IV. The distances AB, CD, and EF are said to be *compatible*, if there exist the points, X, Y, and Z, such that XY is the same as AB, YZ is the same as CD, and XZ is the same as EF.



This notion of compatibility is the only one in the present set that is not familiar to common sense. We do not absolutely need it here, and it is introduced only because it enables us to define in relatively simple terms what is meant by the sum of two distances.

V. The *sum* of the distances AB and CD is a distance compatible with them, and greater than any other such distance if such there be.



In the figure let BV, BW, BX, BY, and BZ all be the same as CD. Then (by Definition IV) AV, AW, AX, AY, and AZ are all compatible with AB and CD. One of the five, namely AX, is greater than any of the others; and it is, in fact, greater than any other distance compatible with AB and CD. We call it the sum of AB and CD.

Here we must make the same remark about the use of the term "sum" that was made above with regard to the term "greater than." From arithmetic we know what the "sum" of two real numbers means. Metrical geometry has to show that the sum of two distances has the same formal properties that belong to the sum of two real numbers. This is very easily accomplished.

VI. If A, B, and C are distinct, and AC is the sum of AB and BC, B is said to be *between* A and C.

VII. If either A is between B and C, or B is between A and C, or C is between A and B; A, B, and C are said to be *collinear.*

VIII. The *straight line* AB is the class of points collinear with A and B, together with A and B.

Attention must now be called to a few characteristics of this series of definitions. In the first place, the definitions are exceedingly simple. If the reader will glance over the whole series of eight, he will see that they follow one upon another in the most natural and obvious fashion. When once the proper indefinables are chosen, the definitions are almost inevitable—not because others to take their place can not be found,[11] but because it would require a good deal of ingenuity or of industry to think of them, while these fairly suggest themselves. The only feature that smacks in the least of artificiality is the notion of "compatibility," and that only on account of the novel term that is used. The problem was, how to combine two distances so as to get their sum. Is not the obvious answer that the outside points shall be as far apart as possible? At any rate, if the definitions are not inevitable, they are at least easy—so easy that the whole set can be grasped by the mind almost without effort, in a single movement of the attention.

If the reader has studied projective geometry, and in particular the projective definition of distance, he will not hesitate to admit that the metrical definition of collinearity is incomparably simpler.

[11] An example of an alternative method is suggested by Professor C. V. Huntington, in *Mathematische Annalen*, Vol. 73, pp. 529f. Professor Huntington takes as his indefinables "sphere" and "is contained in." A "point" is defined as a sphere in which no other sphere is contained. "The point B lies between the points A and C," is then explained as meaning: No sphere X exists, in which A and C are contained, but not B.

In terms of our own indefinables, a sphere is a class of points whose distance from a certain point is not greater than a certain distance. For one sphere to be contained in another, means that every point of the former is a point of the latter. Accordingly, Professor Huntington's definition of the between-relation becomes: No point X exists such that there exists a distance YZ, such that BX is greater than YZ, and neither AX nor CX is greater than YZ, This is obviously more complex than is necessary; so we may put it: No X exists, such that BX is greater than both AX and CX.

Professor Huntington's device suggests others that to him, with his peculiar indefinables, were not available. For example, "B is between A and C" may be explained: If X exists such that AX is the same as CX, this distance is greater than BX. Or we may first explain "A, B, and C are collinear" as meaning: No X exists such that AX, BX, and CX are the same. And we may then distinguish the outer points, say by their greater distance from each other.

In any case, we may proceed to explain "AB is the sum of CD and EF" as meaning: X exists such that it is between A and B, and AX is the same as CD, and BX is the same as EF.

In the second place, the definitions imply nothing as to the number of dimensions in space. Space may equally well have one, two, three, or more dimensions, and these definitions will still hold good. Substitute the word "instant" for "point," and "interval" for "distance," and drop the last two definitions as superfluous, and the account applies perfectly to time. The like is true of the metrical principles which we shall shortly have to consider: they are all equally applicable to space of any number of dimensions.

It has sometimes been held by mathematicians that projective, descriptive, and metrical geometry presuppose one another in that order: that projective geometry is logically prior to descriptive geometry, and both of these to metrical geometry. Logicians, of course, are not at present inclined to take much stock in the notion of logical priority, except as it is conceived with reference to some particular deductive system. It has been found that there is a surprising amount of freedom in the choice of indefinable terms and indemonstrable propositions—how much we do not know. What is logically prior in one construction may be logically posterior in another. Of an absolute order of priority we know nothing.

But for our present purpose, since we have in view an inquiry into empirical foundations, the really important question of priority as between the three types of geometry is as to which gives the simplest conception of the nature of space. In this respect, metrical geometry has an incontestable and enormous advantage. The direction taken by the history of the science is here the path of least resistance. Starting from the theory of spatial measurement, we find spatial order comparatively easy to understand. Starting from the arrangement of points in lines, we find spatial measurement an intrinsically and unavoidably abstruse subject.

Our first task is accomplished. We have seen what, in metrical terms, a linear order means. But the meaning of our assumed indefinables is itself set forth in the postulates which we adopt, and becomes more and more explicit as the consequences of these postulates are successively unfolded, that is to say, as the system of geometry is constructed. It is far beyond the purpose of the present study to attempt the construction of even the foundations of a geometry. It is, indeed, not difficult to set forth a list of principles—only about sixteen are needed—from which the Euclidean geometry can be deduced. But to provide and insure the mutual independence of the principles, so that they may serve as the postulates of a mathematical system, is a task calling for a very special competence and training.

However, I have thought it well to present in this place a state-

ment of certain metrical principles, which, so far as they go, are independent of one another, and which are sufficient as premises for the demonstration of the most important properties of the between-relation. I trust that they will be of service in throwing light upon the empirical considerations that are to follow.[12]

Points, as before, are denoted by capital letters, and different letters need not indicate distinct points.

1. If A and B are farther apart than C and D, C and D are not farther apart than A and B.

In other words, "to be farther apart than" is an asymmetrical relation.

II. If A and B are farther apart than C and D, and E and F are not farther apart than C and D, A and B are farther apart than E and F.

In combination with the preceding, this assures us that if A and B are farther apart than C and D, and if C and D are farther apart than E and F, A and B are farther apart than E and F. That is to say, "to be farther apart than" is a *transitive* relation.

It also follows that the relation "to be just as far apart as" is transitive.

III. If A is not identical with B, then, for all values of X, the distances AB and XX are distinct.

This is used in demonstrating the proposition that the distance XX is constant, and is in fact the *zero-distance;* that is to say, that if XX be added to any distance YZ, the sum is YZ.

IV. If A, B, C, and D be each any point, X exists such that BX is identical with CD, and AX is the sum of AB and CD.

This assures us that any distance and any distance have a sum. It also assures us of a part of what we mean by the "uniformity" of space.

V. If AC is greater than AD, and BC is the sum of BA and AC, BC is greater than BD.

From this it follows that if AC is greater than AD, and XY is any distance, the sum of AC and XY is greater than the sum of AD and XY.

VI. If A and B are not identical, and if BC is not greater than BD; and if AD is the sum of AB and BD, and if AC is the sum of AB and BC; then BD is the sum of BC and CD.

This provides that any two distances shall have a difference. The last three principles, or their equivalents, are necessary to show that

[12] It should be remembered that we assume from general logic that the relation of co-membership in a pair is symmetrical with respect to the two members. Accordingly we assume without question that "the point A and the point B" means the same pair as "the point B and the point A."

the relation "to be greater than" is strictly analogous to the greater-than relation between real numbers.

Assuming these six principles to be true, we can easily prove the associative law for the addition of distances. (The commutative law is in this system a mere identity.) We can also prove the following series of propositions (using the expression (XYZ) to mean "Y is between X and Z"):

(i) If (ABX) and (ABY), and BX is identical with BY, X is identical with Y.

(ii) If (ABX) and (ABY), and BY is greater than BX, then (BXY) and (AXY).

(iii) If (AXB) and (AYB), and AX is identical with AY, X is identical with Y.

(iv) If (AXB) and (AYB), and AY is greater than AX, then (AXY) and (XYB).

(v) If (XAB) and (ABY), then (XAY) and (XBY).

(vi) If (AXB) and (ABY), then (AXY) and (XBY).

And from these six propositions we may derive the following generalization, which establishes the complete determination of the straight line by any two of its points:

If A, B, C, and D are distinct, and if A, B, and C are collinear and A, B, and D are collinear, then A, C, and D are collinear.

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A METAPHYSICIAN'S *PETITIO*

OF those who interest themselves to-day in the general problem of the method of experience, the greater number approach the subject in the light of experience alone. That is to say, they criticize experience without *a priori* presuppositions as to its nature. For example, they observe the processes of scientific investigation as a parent observes the breathing of her child or a physiologist the functions of the brain. They examine the industrial and commercial activities of human beings, their political strategies and recreative sports, with the meticulous care of a psychologist interested in the phenomenon of fatigue. The artistic tastes, the religious rituals and the poetic creations of people are as interesting to the methodologist as the process of metabolism to a biologist. Above all he hugs to his bosom the transformations of attitude and emotion through which reflective life passes out of one and into another of these phases of experience, believing that therein especially is ultimately to be found the synthesis that endows civilization